Journal of Geometry and Physics 39 (2001) 129–134

JOURNAL OF GEOMETRY AND PHYSICS

Flat connections on a punctured sphere and geodesic polygons in a Lie group

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Received 25 September 2000

Abstract

Let G be a compact connected Lie group and X denote the complement of n distinct points of the sphere S^2. The space of isomorphism classes of flat G connections on X with fixed conjugacy class of holonomy around each of n punctures has a natural symplectic structure. This space is related to the space of geodesic n-gons in G. The space of geodesic polygons in G has a natural 2-form. It is shown that this 2-form coincides with symplectic form on the space of isomorphism classes of flat G-connections on X satisfying holonomy condition at the punctures. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 53C05; 58F05

Subj. Class.: Differential geometry

Keywords: Connections; Punctured sphere; Geodesic polygons; Lie group

1. Introduction

Fix n distinct points

$$S := \{s_1, s_2, \ldots, s_n\} \subset S^2$$

of the two-dimensional sphere with $n \geq 2$. The complement $S^2 \setminus S$ will be denoted by X. Fix a point $s' \in X$. The fundamental group $\pi_1(X, s')$ will be denoted by Γ. Note that Γ is the quotient of the free group generated by n elements, say $\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n}$, by the normal subgroup generated by $\gamma_{s_1}\gamma_{s_2}\cdots\gamma_{s_n}$. The free homotopy class represented by any γ_{s_i} is the clockwise loop around s_i.

Let G be a compact connected Lie group. The Lie algebra of G will be denoted by $\mathfrak{g}$.The group G acts on itself by inner conjugation, and the infinitesimal form of this action is the

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PII: S0393-0440(01)00004-3

adjoint action of G on $\mathfrak{g}$. Fix a G-invariant symmetric bilinear form B on $\mathfrak{g}$ which is positive definite. So, if G is semisimple, then B could be the negative of the Killing form.

For each $s \in S$, fix a conjugacy class $C_s \subset \mathfrak{g}$. In other words, each C_s is an orbit of the G-action. They need not be distinct.

Let $\bar{A}$ denote the space of all maps f from the finite set S to $\mathfrak{g}$ such that $f(s) \in C_s$ for each $s \in S$ and also satisfying the condition

$$\exp(f(s_1))\exp(f(s_2))\cdots\exp(f(s_{n-1}))\exp(f(s_n)) = e,$$

where $\exp : \mathfrak{g} \to G$ is the exponential map and e is the identity element. The group G acts on $\bar{A}$ through inner conjugation. In other words, the action of $g \in G$ on f sends it to the composition

$$S \xrightarrow{f} \mathfrak{g} \xrightarrow{g} \mathfrak{g}.$$

Let

$$\psi : \bar{A} \to \mathcal{A} := \frac{\bar{A}}{G} \tag{1.1}$$

denote the quotient map.

Note that for any f as above, sending each $s \in S$ to $\exp(f(s))$ we get a homomorphism from Γ to G. This in turn gives a flat G-bundle over X. Let

$$\mathcal{R} \subset \frac{\mathrm{Hom}(\Gamma, G)}{G}$$

denote the subset consisting of homomorphisms ρ such that $\rho(\gamma_s) \in \exp(C_s)$ for all $s \in S$. Therefore, we have a map

$$H : \mathcal{A} \to \mathcal{R} \tag{1.2}$$

defined by $H(f)(s) = \exp(f(s))$. From the properties of the exponential map it follows that H is a diffeomorphism on an open dense subset of $\mathcal{A}$.

Using B, the space $\mathcal{R}$ has a natural symplectic structure [1,2,6,7]. The construction of this symplectic form, which we will denote by Ω_0, will be recalled later. Let

$$\Omega := H^*\Omega_0 \tag{1.3}$$

be the closed 2-form on $\mathcal{A}$.

Using translations in G, the bilinear form B defines a Riemannian metric on G, which will be denoted by $\bar{B}$. The condition that B is G-invariant ensures that $\bar{B}$ does not depend on whether the left or the right translation is used in its definition.

Given any $f \in \bar{A}$, we construct a geodesic polygon in G with the vertices

$$f_i := \exp(f(s_1))\exp(f(s_2))\cdots\exp(f(s_i)),$$

where $i \in [1, n]$. Note that $f_n = e$. Setting, $f_0 = e$ for $i \in [1, n]$, the edge connecting the edge f_{i-1} to f_i is the geodesic segment defined by

$$t \mapsto f_{i-1}\exp(tf(s_i)),$$

where $t \in [0, 1]$.

A *geodesic n-gon* in G is a continuous map

$$\tau : [1, n] \to G \tag{1.4}$$

such that $\tau(0) = e = \tau(n)$ and the restriction of τ to each interval $[i, i+1]$, where $i \in \{0, 1, \ldots, n-1\}$, is a geodesic segment. The space of polygons has drawn considerable attention in recent times [9,10].

Denoting the space of all geodesic n-gons by $\mathcal{G}$, we have a smooth map

$$\phi : \bar{A} \to \mathcal{G} \tag{1.5}$$

that sends any f to the geodesic polygon defined above.

Consider the tri-linear form θ_0 on $\mathfrak{g}$ defined by $(a, b, c) \mapsto B(a, [b, c])$. This form is clearly alternating and G invariant. Let $\bar{\theta}$ denote the 3-form on G obtained from θ_0 using translations. Since θ_0 is G-invariant, left and right translations of it coincide. It is known that $\bar{\theta}$ is closed.

Let $\tau : [1, n] \to G$, as in (1.4), be a geodesic n-gon and let $l_1, l_2, \ldots, l_n$ be its edges. A tangent vector $v \in T_\tau \mathcal{G}$ gives a *Jacobi field* along each l_i. Recall that a Jacobi field along a geodesic segment l is a Riemannian manifold M is a smooth section of the restricted tangent bundle $TM|_l$ over l which is obtained from a 1-parameter family of geodesic segments deforming l [4], [5, pp. 357]. From the definition of a Jacobi field it is immediate that v gives a Jacobi field on each l_i. In particular, it is a section of $TG|_\tau$ which is piece-wise smooth. We will call a section of τ^*TG, whose restriction to each edge is a Jacobi field, *as a Jacobi field on* τ.

Now, given an ordered pair of tangent vector $v, w \in T_\tau \mathcal{G}$, let $\bar{v}$ and $\bar{w}$ denote the corresponding sections of τ^*TG defining Jacobi fields on τ. Recall that $\tau^*\bar{\theta}$ is a section of $\tau^* \wedge^3 T^*G$. Therefore, contracting $\tau^*\bar{\theta}$ with $\bar{v}$ and $\bar{w}$ we get a piece-wise smooth 1-form on $[1, n]$. Let $\theta(v, w) \in \mathbb{R}$ be the integral of this 1-form on $[1, n]$.

Consequently, we have a 2-form

$$\theta \in C^\infty\left(\mathcal{G}, \bigwedge^2 T^*\mathcal{G}\right) \tag{1.6}$$

that sends the ordered pair of tangent vectors v, w to $\theta(v, w)$ defined above.

Now we are in a position to state the result obtained here.

Theorem 1.1. *The* 2*-form* $\psi^*\Omega$ *on* $\bar{A}$ *coincides with* $\phi^*\theta$, *where* ψ, Ω *and* ϕ *are defined in* (1.1), (1.3) *and* (1.5), *respectively. Equivalently,* $\phi^*\theta$ *descends to* $\mathcal{A}$ *and the descended form coincides with* Ω.

The proof will be carried out in the next section.

2. Equivalence of symplectic forms

We start by recalling the description of the symplectic form Ω_0.

The space $\mathcal{R}$ defined in the introduction is clearly the space of isomorphism classes of all pairs (P, ∇), where P is a principal G-bundle over X and ∇ is a flat connection on P, with the property that for any $s \in S$, the holonomy of the connection ∇ around s lies in the conjugacy class in G defined $\exp(C_s)$. Let $\mathrm{ad}(P)$ denote the adjoint bundle associated to P for the adjoint representation of G in $\mathfrak{g}$. The connection ∇ induces a flat connection on $\mathrm{ad}(P)$. Let $\underline{\mathrm{ad}(P)}$ denote the local system on X given by the sheaf of flat sections of $\mathrm{ad}(P)$.

Let $\iota : X \to S^2$ denote the inclusion map.

The tangent space to $\mathcal{R}$ at the point $\rho = (P, \nabla)$ has the following description [2, Lemma 2.2], [3, Proposition 2.4]

$$T_\rho \mathcal{R} = H^1(S^2, \iota_* \underline{\mathrm{ad}(P)}).$$

The bilinear form B on $\mathfrak{g}$, using its G-invariance condition, induces a bilinear form on the fibers of $\underline{\mathrm{ad}(P)}$. The symplectic form Ω_0 on $\mathcal{R}$ is defined by the following pairing [2, Theorem 2.8], [3, (2.5)]

$$\begin{aligned} T_\rho \mathcal{R} \otimes T_\rho \mathcal{R} &= H^1(S^2, \iota_* \underline{\mathrm{ad}(P)}) \otimes H^1(S^2, \iota_* \underline{\mathrm{ad}(P)}) \\ &\to H^2(S^2, \iota_* \underline{\mathrm{ad}(P)} \otimes \iota_* \underline{\mathrm{ad}(P)}) \xrightarrow{\tilde{B}} H^2(S^2, \mathbb{R}) = \mathbb{R}. \end{aligned}$$

Since B is G-invariant, it defines a bilinear pairing $\tilde{B} : \iota_* \underline{\mathrm{ad}(P)} \otimes \iota_* \underline{\mathrm{ad}(P)} \to \mathbb{R}$.

The symplectic structure Ω_0 has other alternative descriptions [8].

The pulled back form $\psi^* \Omega = (H \circ \psi)^* \Omega_0$ on $\bar{A}$ will be described. The maps H and ψ are defined in (1.2) and (1.1), respectively. For this purpose it is necessary to describe the tangent bundle of $\bar{A}$.

Let $f : S \to \mathfrak{g}$ be an element of $\bar{A}$. So, $f(s) \in C_s$ for all $s \in S$ and $\prod_{i=1}^n f(s_i) = e$. For any $i \in [1, n]$, set

$$f_i := \exp(f(s_1)) \exp(f(s_2)) \cdots \exp(f(s_i)) = \prod_{j=1}^{i} \exp(f(s_j)) \in G. \tag{2.1}$$

So, $f_n = e$. For any $s \in S$, let

$$\mathfrak{g}(f(s)) := [f(s), \mathfrak{g}] \subset \mathfrak{g} \tag{2.2}$$

be the linear subspace of $\mathfrak{g}$. Note that $\mathfrak{g}(f(s)) = 0$ if $f(s)$ is in the center of $\mathfrak{g}$.

As before, the adjoint action of G on $\mathfrak{g}$ will be denoted by ad. So, $g \in G$ takes $\alpha \in \mathfrak{g}$ to $\mathrm{ad}(g)(\alpha)$.

Let V_f denote the space of all functions $v : S \to \mathfrak{g}$ satisfying the following two conditions:

1. $v(s) \in \mathfrak{g}(f(s))$ for every $s \in S$, where $\mathfrak{g}(f(s))$ is defined in (2.2);
2. for the adjoint action of $f_i \in G$ on $v(s_i)$

$$\sum_{i=1}^{n} \mathrm{ad}(f_i)(v(s_i)) = 0,$$

where f_i is defined in (2.1).

Note that V_f is a linear subspace of the direct sum $\mathfrak{g}(S) := \oplus_{s\in S}\mathfrak{g}$ of n copies of $\mathfrak{g}$.

Lemma 2.1. *The tangent space to $\bar{A}$ at f, namely $T_f\bar{A}$, is canonically identified with the vector space V_f.*

Proof. The space $\bar{A}$ is a subset of $\mathfrak{g}(S) := \oplus_{s\in S}\mathfrak{g}$, the space of all functions from S to $\mathfrak{g}$. Since, $\mathfrak{g}(S)$ is a vector space, its tangent space at any point is identified with the vector space $\mathfrak{g}(S)$ itself. Therefore, for any $f \in \bar{A}$ and $v \in \mathfrak{g}(S)$, we need to determine the conditions that ensure that $v \in T_f\bar{A}$. In other words, we need to find the infinitesimal form of the conditions that define the subset $\bar{A}$ of $\mathfrak{g}(S)$.

The first of the two conditions defining $\bar{A}$ says that $h \in \bar{A}$ implies $h(s) \in C_s$ for all $s \in S$. The infinitesimal form of this condition is easily seen to be the condition that $v(s) \in \mathfrak{g}(f(s))$ for all $s \in S$. Indeed, this is an immediate consequence of the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{ad}(\exp(t\alpha))\beta = [\alpha, \beta]$$

valid for any $\alpha, \beta \in \mathfrak{g}$.

The other condition defining $\bar{A}$ says that $h \in \bar{A}$ implies $\prod_{s\in S} h(s) = e$. The adjoint action and the exponential map commute, namely $\exp(\mathrm{ad}(g)\alpha) = g\exp(\alpha)g^{-1}$ for any $g \in G$ and $\alpha \in \mathfrak{g}$. Using this it is easy to deduce that the infinitesimal form of the second condition says that $\sum_{i=1}^{n}\mathrm{ad}(f_i)(v(s_i)) = 0$. This completes the proof of the lemma. □

We will now use a description of Ω_0 given in [1, pp. 109, Theorem 1].

Take a point $f \in \bar{A}$. Take two tangent vectors $v, w \in V_f = T_f\bar{A}$. So, in particular, f, v and w are functions from S to $\mathfrak{g}$. The following proposition is a direct consequence of [1, Theorem 1].

Proposition 2.2. *The number*

$$\sum_{s\in S} B(f(s), [v(s), w(s)]) \in \mathbb{R}$$

coincides with $\psi^*\Omega(v, w)$, where $\psi^*\Omega$ is the 2*-form on $\bar{A}$ with ψ and Ω defined in* (1.1) *and* (1.3), *respectively.*

Take any $g \in G$ and $\alpha \in \mathfrak{g}$. Consider the geodesic segment

$$l : [0, 1] \to G$$

defined by $t \mapsto g\exp(t\alpha)$. For any $\beta \in \mathfrak{g}$, let $\hat{\beta}$ denote the right invariant vector field on G such that $\hat{\beta}(e) = \beta$. The section of l^*TG obtained by restricting $\hat{\beta}$ to the image of l will be denoted by $\bar{\beta}$.

It is easy to see that $\bar{\beta}$ is a Jacobi field on l. Indeed, for any $c \in \mathbb{R}$ consider the geodesic segment

$$l_c : [0, 1] \to G$$

defined by $t \mapsto \exp(c\beta)g\exp(t\alpha)$. Clearly, $l_0 = l$, and the tangent space to l defined by the derivative $(\mathrm{d}l_c/\mathrm{d}c)|_{c=0}$ coincides with $\bar{\beta}$.

Recall the 3-form $\bar{\theta}$ on G constructed in Section 1 from the tri-linear form θ_0 on $\mathfrak{g}$. Given any ordered pair of vectors $\beta, \gamma \in \mathfrak{g}$, the contraction of $\bar{\theta}$ by the ordered pair $\bar{\beta}, \bar{\gamma}$, defines a 1-form on [0, 1]. This 1-form on [0, 1] will be denoted by $\omega(\beta, \gamma)$.

In view of Proposition 2.2 and the definition of the 2-form θ in (1.6), the proof of Theorem 1.1 is completed by the following proposition.

Proposition 2.3. *Given any pair* $\beta, \gamma \in \mathfrak{g}$,

$$\int_{[0,1]} \omega(\beta, \gamma) = \theta_0(\alpha, \beta, \gamma) := B(\alpha, [\beta, \gamma]) \in \mathbb{R},$$

where $\omega(\beta, \gamma)$ *is defined above*.

Proof. It is easy to check that the 1-form $\omega(\beta, \gamma)$ on [0, 1] is the constant form that sends a unit tangent vector to $B(\alpha, [\beta, \gamma])$. From this the proposition follows immediately. □

Given any geodesic polygon τ as in (1.4), using Proposition 2.3 for each individual edge of τ it follows that $\phi^*\theta$ coincides with the expression for $\psi^*\Omega$ given by Proposition 2.2. This completes the proof of Theorem 1.1.

References

[1] A.Y. Alekseev, A.Z. Malkin, Symplectic structure of the moduli space of flat connections on a Riemann surface, Commun. Math. Phys. 169 (1995) 99–119.
[2] I. Biswas, K. Guruprasad, Principal bundles on open surfaces and invariant functions on Lie groups, Int. J. Math. 4 (1993) 535–544.
[3] I. Biswas, N. Raghavendra, Curvature of the determinant bundle and the Kähler form over the moduli of parabolic bundles for a family of pointed curves, Asian J. Math. 2 (1998) 303–324.
[4] J. Cheeger, D.G. Ebin, Comparison Theorems in Riemannian Geometry, North-Holland, Amsterdam, 1975.
[5] M.P. Do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, NJ, 1976.
[6] W.M. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. Math. 54 (1984) 200–225.
[7] W.M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surfaces, Inv. Math. 85 (1986) 263–302.
[8] K. Guruprasad, J. Huebschmann, L. Jeffrey, A. Weinstein, Group systems, groupoids, and moduli spaces of parabolic bundles, Duke Math. J. 89 (1997) 377–412.
[9] M. Kapovich, J. Millson, On the moduli space of polygons in the Euclidean plane, J. Diff. Geom. 42 (1995) 133–164.
[10] M. Kapovich, J. Millson, The symplectic geometry of polygons in Euclidean space, J. Diff. Geom. 44 (1996) 479–513.